FIBONACCI *KOLAMS* -- AN OVERVIEW

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This paper is an overview of all my work on Fibonacci *Kolam*s as of end of the year 2015 that is included in my website

www.vindhiya.com/snaranan/fk/index.htm

1. Introduction.

*Kolam*s are decorative designs of line drawings around dots in a regular grid as a template. Known as '*Pulli Kolam*s' they adorn entrances of households and temples, especially in South India. The folk-art is handed down through generations of women from historic times, nurtured by housewives and housemaids in rural as well as urban areas. The designs are intricate, complex and creative, constrained only by some broad rules. A common rule is four-fold rotational symmetry: viewed from all the four sides North, East, South and West, the *kolam* appears the same. The *kolam*s generally have multiple loops (closed curves) but single-loop *kolam*s are special and are harder to achieve.

The simplest geometrical shape with four-fold symmetry is the square. More complex shapes can be broken down into square modules. Rectangles can have only two-fold rotational symmetry. Rectangular shapes serve as modules in building larger squares. Therefore we consider only square and rectangular *kolam*s. We shall see that the mathematical tool 'Fibonacci Recurrence' is a natural basis for generating square and rectangular *kolam*s of any desired size.

2. Fibonacci Recurrence.

In Fibonacci Recurrence, a series of integers is generated in which every integer is a sum of the two preceding integers. Given two seed numbers, all the subsequent numbers are determined. Such a series is called the Generalized Fibonacci Series. The simplest such series is the canonical Fibonacci Series starting with 0, 1:

0 1 1 2 3 5 8 13 21 34 55 89 ....

If the starting numbers are 2, 1 we have the Lucas Series

2 1 3 4 7 11 18 29 47 76 ....

For *kolam* designs we are interested in a set of four consecutive numbers in the series, e.g. (3 5 8 13), (3 4 7 11). In a Generalized Fibonacci Series a quartet is denoted as Q (a b c d). Because of Fibonacci Recurrence

$$c = a + b \qquad d = b + c = a + 2b \tag{1}$$

An equation relating a b c d is

$$d^2 = a^2 + 4\,b\,c \tag{2}$$

This identity is the basis for Fibonacci *kolam*s. It has a geometrical counterpart: In Figure 1, a square d^2 has a smaller concentric square a^2 and the space between the two is filled up with four rectangles (b x c) placed in a cyclical pattern. This construction is the essence of Fibonacci *kolam*s. It ensures four-fold symmetry. The rectangles are identical and need have no symmetry property, but the central square a^2 should have four-fold symmetry. In Figure 1 the quartet is Q (2 3 5 8). To generate the 'rectangle b c'

$$\text{'Rectangle } b\,c\text{'} = \text{'Square } b^2\text{'} + \text{'Rectangle } a\,b\text{'} \tag{3}$$

Using these equations, we can build a hierarchy of squares and rectangles leading up to any desired size. Note that the rectangle b c has no two-fold rotational symmetry. These rectangles are called 'Golden Rectangles'. The ratio of the sides approaches a limiting value $\varphi = (1 + \sqrt{5})/2 = 1.61803.....$ called the Golden Ratio. In geometry it is related to pentagons, decagons and the 3-dimensional Platonic solids. It is pervasive in Nature; in branching of trees, arrangement of flower petals, seeds and leaves and spiral shapes of florets in sun flowers and in sea-shells. It is closely related to growth processes in Nature. The Golden Ratio is also believed to figure prominently in Western art and architecture. It is a key feature of Fibonacci *Kolam*s.

3. Square *Kolam*s.

In Part I, *Kolam* designs were based on the canonical Fibonacci Series 0 1 1 2 3 5 8 13 21 34 ....[1]. Using equations (2) and (3) a sequence of square *kolam*s 3^2 5^2 8^2 13^2 21^2 can be built in a hierarchical scheme as shown in 'Skeleton Diagrams' in Figure 2. Actual square *kolam*s 3^2 5^2 8^2 13^2 21^2 and rectangular *kolam*s (2 x 3, 3 x 5, 5 x 8, 8 x 13) were presented in Part I. In Figure 3 is shown the 8^2 *kolam* based on the quartet Q (2 3 5 8) featured in the skeleton diagram of Figure 1. It is reproduced from Figure 5(d) in Part 1.

The *Kolam* designs are governed by some ground rules adopted mainly for aesthetic reasons. They are: four-fold symmetry, single loop and 'no empty four-sided islands'. Fibonacci recurrence ensures four-fold rotational symmetry. The larger square consists of a smaller square and four cyclic rectangles. Assuming the five constituents are all single-loop, the task is to merge them together at splicing points which are symmetrically placed to preserve four-fold symmetry. In general when all the splices are completed, the final outcome can be multiple loops. While most traditional *kolam*s have no requirement of 'single loop', single-loop kolams are regarded as special in aesthetic value. Single loop is achieved by a careful choice of splicing.

In Part II [2], a general scheme to create square and rectangular *kolam*s based on Generalized Fibonacci Series, was given. With the squares 4^2 6^2 7^2 9^2 10^2 and the earlier *kolam*s 3^2 5^2 8^2 all square *kolam*s m^2 (m = 3 to 10) are covered. It is to be noted that 2^2 was first overlooked in [1] but later included in [2]. In fact, it is the smallest non-trivial Fibonacci *kolam* with Q (0 1 1 2) of the canonical Fibonacci series. In Figure 4 the top row (a) (b) (c) shows 3^2 5^2 and 3 x 5 *kolam*s. Middle row (d) (e) (f) shows 2^2 4^2 6^2 *kolam*s. Notice that in the 2^2 'clover-leaf' pattern, there is no central square and four loops touch each other at the centre. The clover-leaf is rare in traditional *kolams*. One such rare occurrence is in the 4^2 in Fig 4(e) known as '*Brahmamudi'* (Brahma's knot). Clover-leaf patterns enhance the overall aesthetics of *kolam*s. In Figure 5 is the 21^2 kolam based on Q (5 8 13 21). The clover-leaf patterns can be seen in the 6^2 of Figure 3(f) and also in the 21^2.

Among rectangles the smallest non-trivial rectangle is 2 x 3. It is obtained by splicing together 2^2 and 1 x 2 rectangle. All the possible variants are discussed in [2]. It turns out that there are six 2 x 3 single-loop *kolam*s, labelled E R G H U S. They are given in the last row of Figure 4 (g). Different orientations – rotations and reflections – give a total of 30 distinct shapes. As constituents of larger *kolam*s different orientations will result in different patterns.

Parts I and II cover all square *kolam*s of any arbitrary size, with four-fold symmetry [1,2]. Rectangular *kolam*s figure mainly as constituents of the squares. They are the 'Golden Rectangles'; as stand-alone rectangles they usually lack any symmetry (reflection or rotation). As constituents they are not required to have any symmetry or be single-loop. For example the '*Pi-kolam*' 7 x 22 is used as a constituent of the 29^2 square *kolam* based on Q (15 7 22 29) in the 'Special *Kolam*s' [3].

4. Rectangles.

Fibonacci Recurrence can be invoked to draw rectangular *kolam*s with two-fold rotational symmetry – the *kolam* looks the same viewed from North and South *or* from East and West. Part III deals with such *kolam*s using two Fibonacci Quartets instead of one used for square *kolam*s [4]. In analogy with equation (1), the pair of quartets is

$$Q_1\ (a_1\ \ b_1\ \ c_1\ \ d_1)$$

$$Q_2\ (a_2\ \ b_2\ \ c_2\ \ d_2)$$

$$c_1 = a_1 + b_1 \qquad d_1 = c_1 + b_1 = a_1 + 2\,b_1$$

$$c_2 = a_2 + b_2 \qquad d_2 = c_2 + b_2 = a_2 + 2\,b_2$$

An identity connecting all the numbers – in analogy to equation (2) is

$$d_1\ d_2 = a_1\ a_2 + 2\,b_1\ c_2 + 2\,c_1\ b_2$$

The geometrical analogue is: a rectangle of sides $d_1\,d_2$ has a concentric rectangle $a_1\,a_2$ and two pairs of rectangles $b_1\,c_2$ and $c_2\,b_1$. In each pair, one rectangle is rotated 180° with respect to the other to ensure two-fold rotational symmetry. In addition, if rectangle a_1a_2 has two-fold symmetry, the overall *kolam* d_1d_2 has two-fold symmetry. The construction is illustrated in Figure 6 using

$$Q_1\ (3\ \ 2\ \ 5\ \ 7)$$

$$Q_2\ (5\ \ 3\ \ 8\ \ 11)$$

As in the case of Fibonacci square *kolam*s, Fibonacci rectangular *kolam*s too can be drawn of any desired size. Many *kolam*s varying in size from 2 x 3 to 7 x 22 (*Pi-kolam*) are given. The *Pi-kolam* has two-fold symmetry unlike its counterpart in [3]. A rectangular *kolam* 9 x13 is reproduced from [4] in Figure 7.

Squares are special cases of rectangles. By setting d_1=d_2 in the two quartets we obtain square *kolam*s with two-fold symmetry (not four-fold symmetry). A lesser symmetry leads to a larger variety of *kolam*s. Square *kolam*s 5^2 to 15^2 are presented in [4]. The 15^2 *kolam* is reproduced in Figure 8.

The choice of splicing points to merge the constituent rectangles determines the number of loops. To obtain a single loop, a good strategy is to go for maximal or near maximal splicing and do all the splices in one attempt. Even after a large number of splices (say 20 or more) the end result is usually a few loops (up to 5); most common are one and three loops. To reduce the number of loops some splices may be removed. Choices of Fibonacci Quartets and splicing points are discussed in detail in [3].

'*Kolam* Slides' is a PowerPoint exposition consisting of two parts: (1) a review of some interesting properties of Fibonacci Numbers and (2) *Kolam* Designs based on Fibonacci Recurrence with excerpts from Parts I and II [5]. It is based on a talk to a group of fresh student trainees of Tata Consultancy Services in Chennai in 2008.

In preparation:

(1) The connection between splices and loops is traced through a study of 'evolution of loops' as a sequence of splices is made. A splice can result in either an increase or a decrease of the number of loops by one. (Although in principle a splice can leave the number of loops unchanged, this happens rarely. The reason seems to be related to the conditions imposed on the choice of splicing points). This observation enables a simple mathematical modelling of evolution of loops as the number of splices is increased. It is found that the number of loops converges to low odd values, mostly one or three. The probabilities of obtaining 1, 3, 5 loops are 0.37, 0.55 and 0.08 respectively. These results are consistent with observations [6].

(2) To explore the full range of variety in *kolam*s, especially of large size, we need the aid of the computer. A major advance towards computer-aided design is the finding that in Fibonacci *Kolam*s the unit cells (squares with dots in the centre) come in eight basic shapes. Different orientations (rotations and reflections) yield 31 distinct shapes. This makes for simple coding of *Kolam* patterns [7].

Acknowledgment: The initiative to create a website for my unpublished works came from my brother S. Srinivasan. 'Fibonacci *Kolam*s' form a major part of the contents and are presented with appropriate annotations by Srinivasan. His daughter Divya Srinivasan helped with the design of the website in initial stages. I am

thankful to both of them for their elegant efforts. I thank my daughter Venil N. Sumantran for useful comments and suggestions.

References.

[1] *Kolam* Designs based on Fibonacci Numbers. Part I. Square and Rectangular *Kolam*s. www.vindhiya.com/snaranan/fk/index.htm (Paper 1).

[2] *Kolam* Designs based on Fibonacci Numbers. Part II. Square and Rectangular designs of arbitrary size based on Generalized Fibonacci Numbers. www.vindhiya.com/snaranan/fk/index.htm (Paper 2).

[3] Special *Kolam* Slides. www.vindhiya.com/snaranan/fk/index.htm

[4] *Kolam* Designs based on Fibonacci Numbers. Part III. Rectangular *Kolam*s with two-fold Rotational Symmetry. www.vindhiya.com/snaranan/fk/index.htm (Paper 3).

[5] *Kolam* Slides. www.vindhiya.com/snaranan/fk/index.htm.

[6] [7] In Preparation.

Figures.

Chennai, 31 December 2015 *S. Naranan*

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Figure 1: Construction of Square Fibonacci Kolams

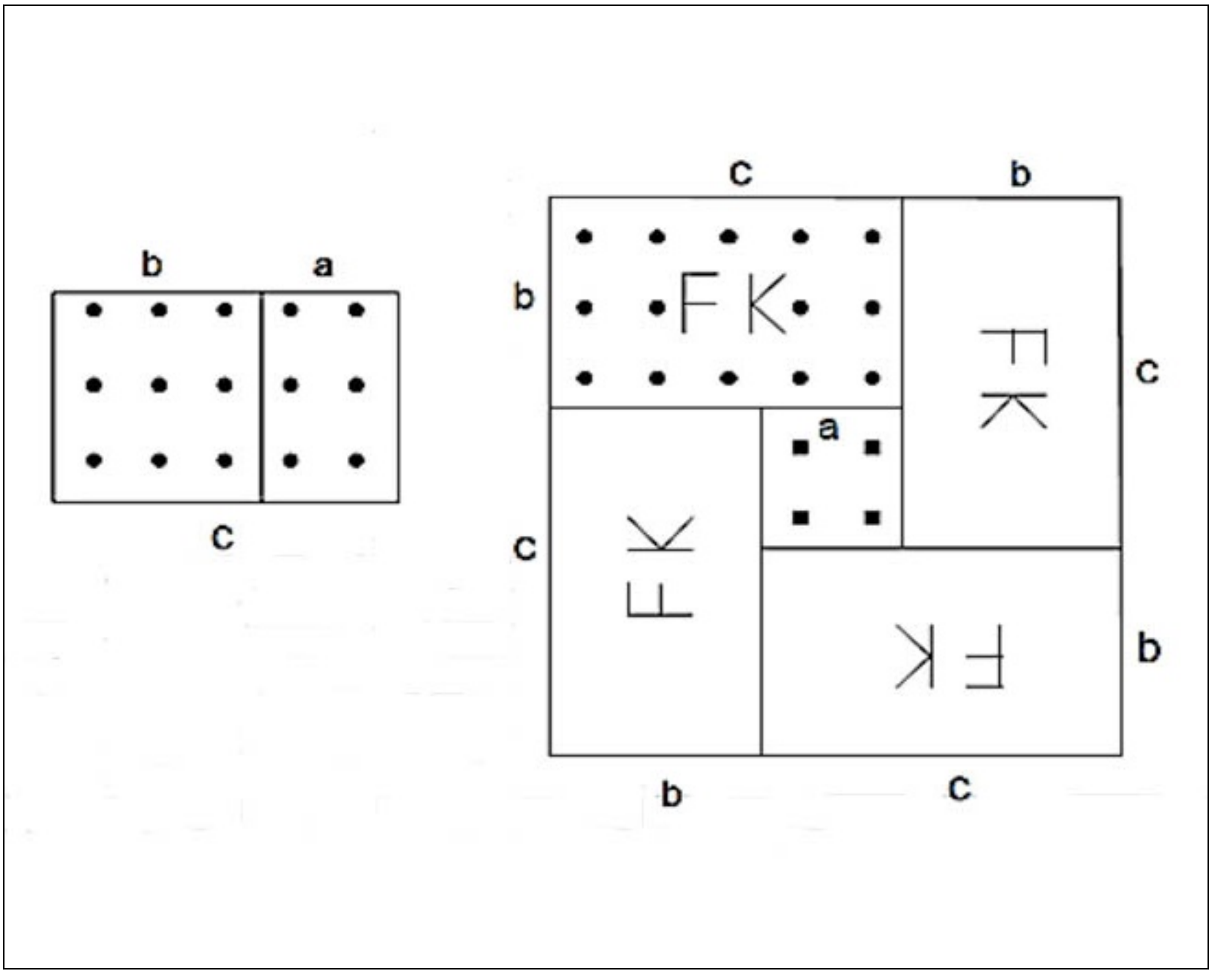

Figure 2: Fibonacci *Kolams* – Skeleton Diagram

FIBONACCI KOLAMS -- SKELETON DIAGRAMS

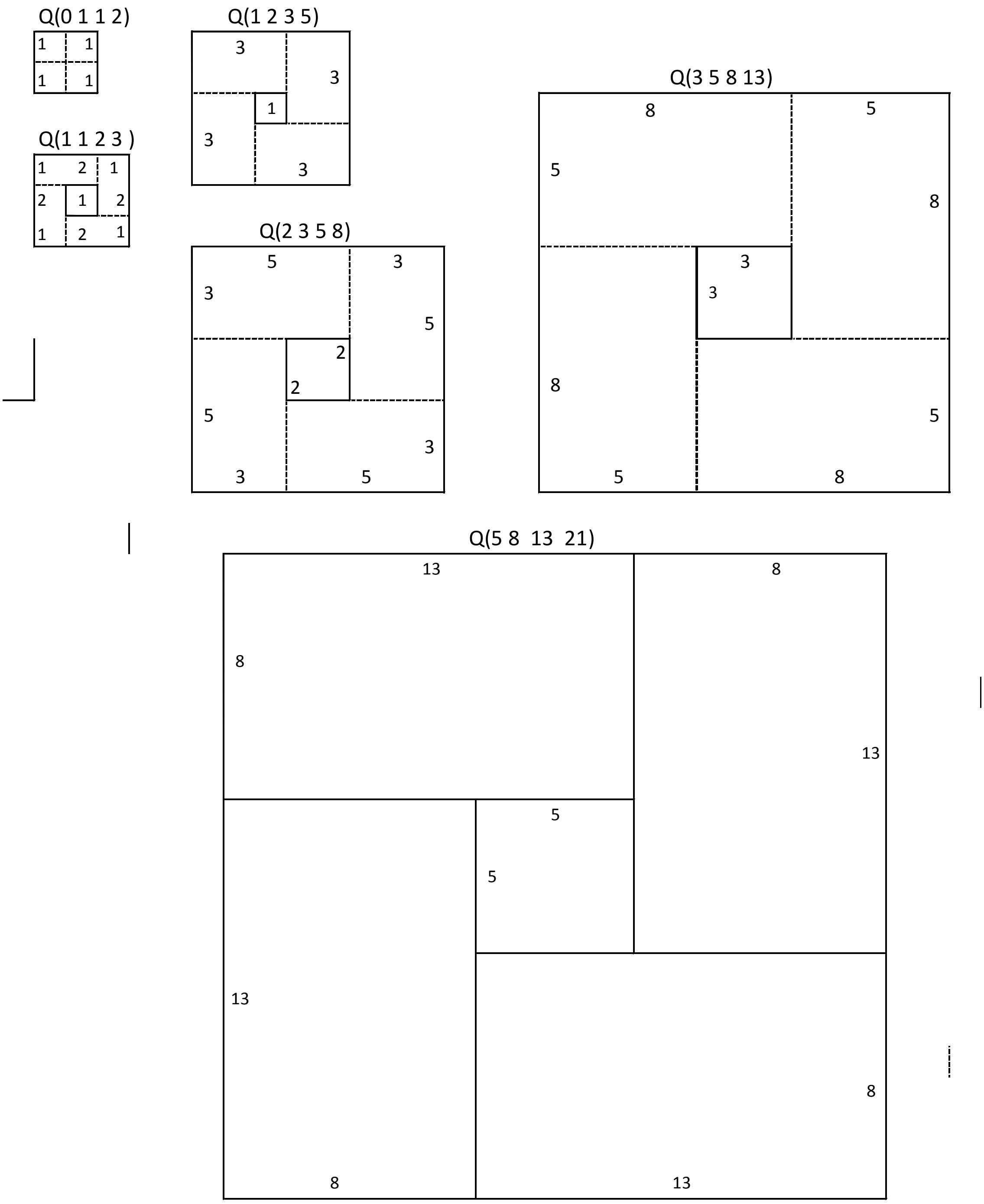

Figure 3: 8^2

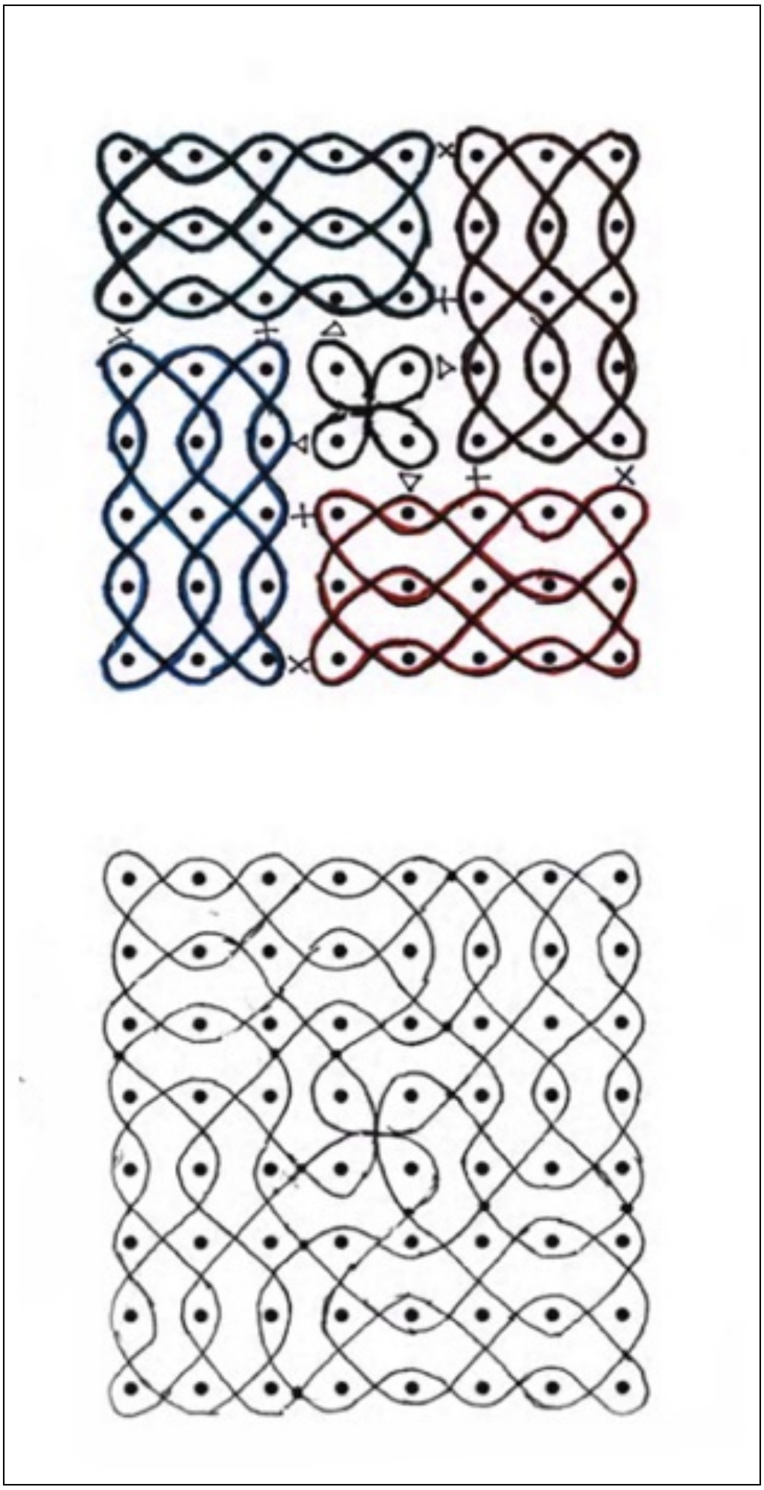

Figure 4: Illustration of Square and Rectangular *Kolams*

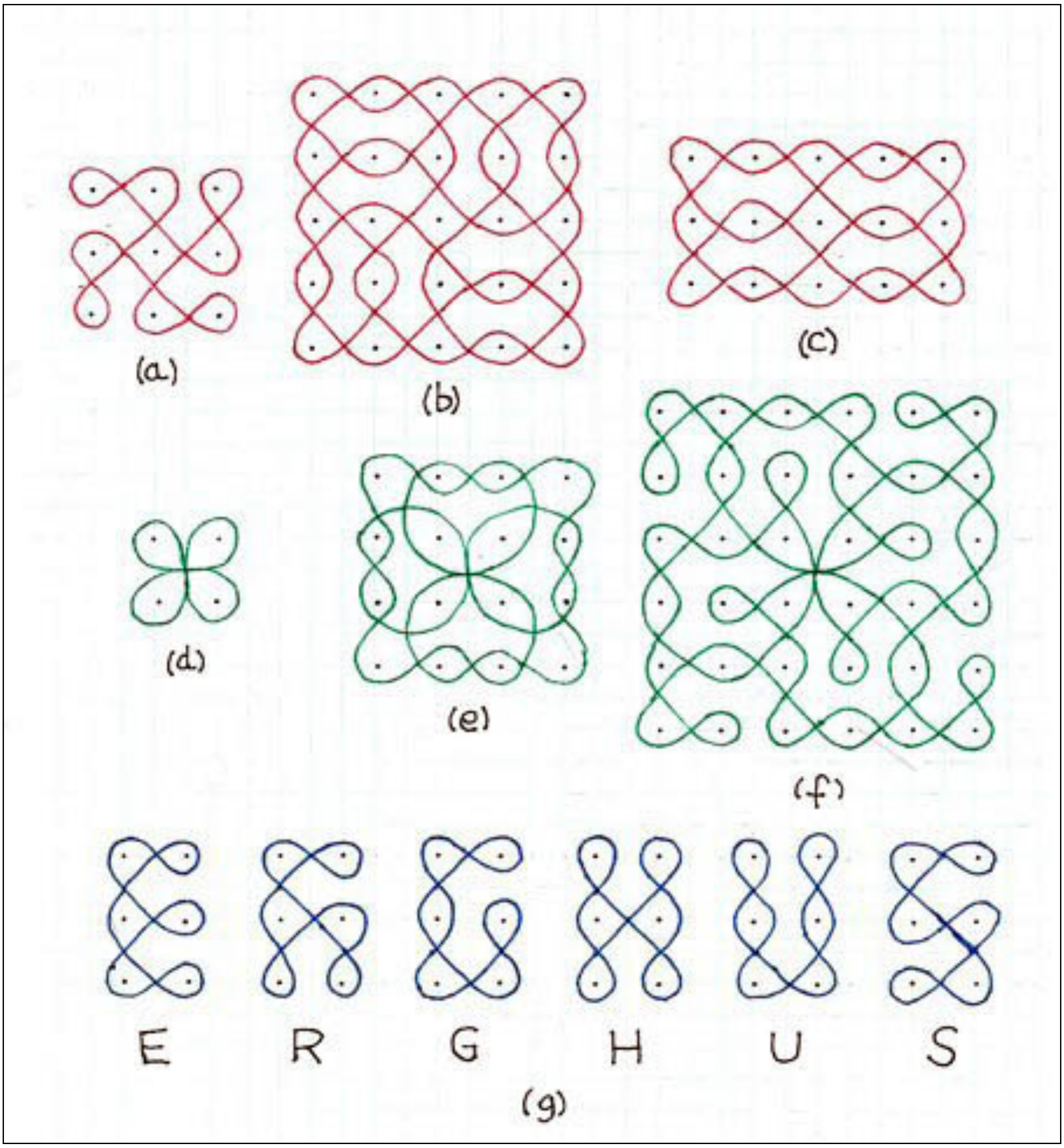

Figure 5: 21^2

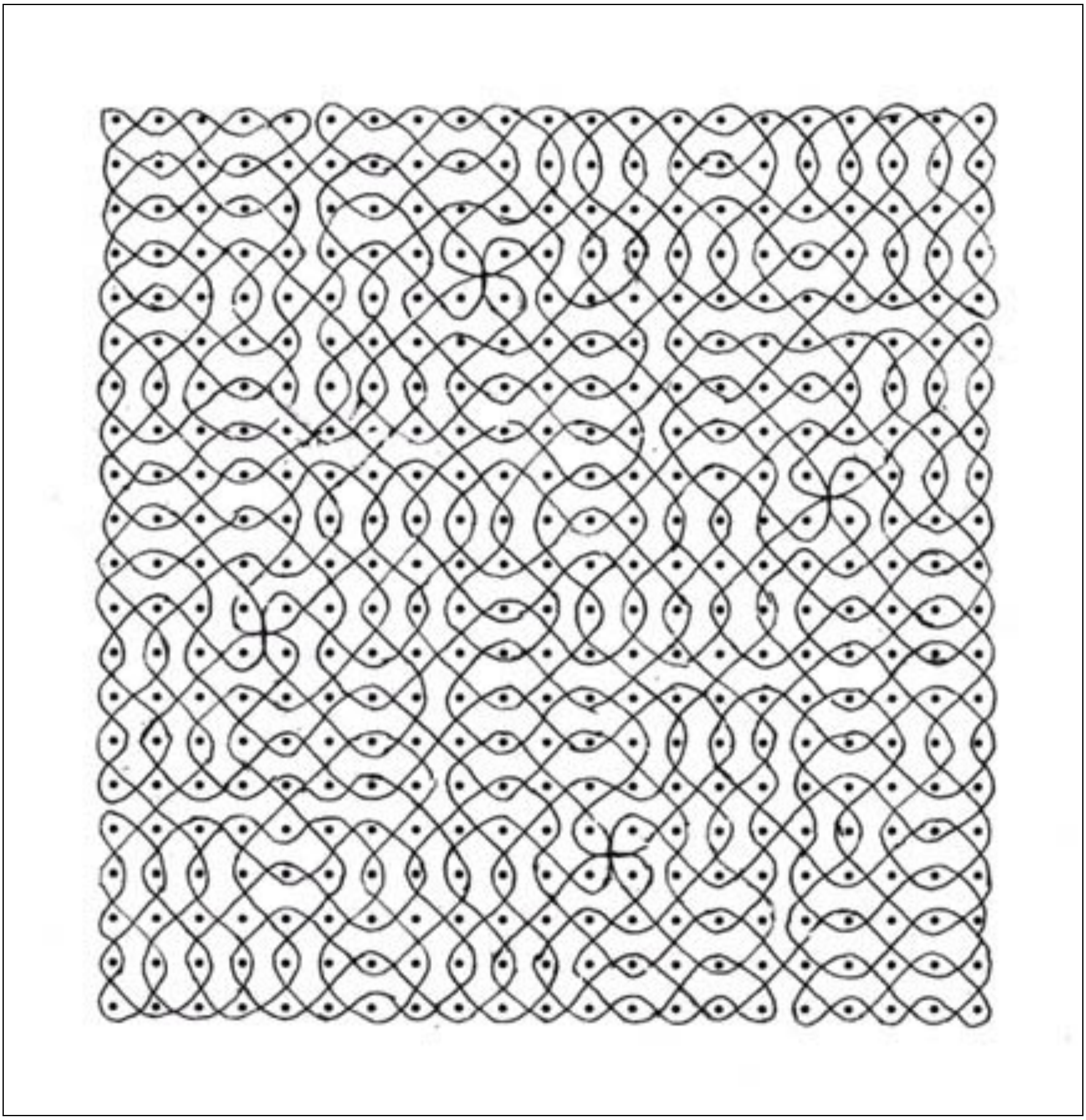

Figure 6: Construction of Rectangular Fibonacci *Kolam*

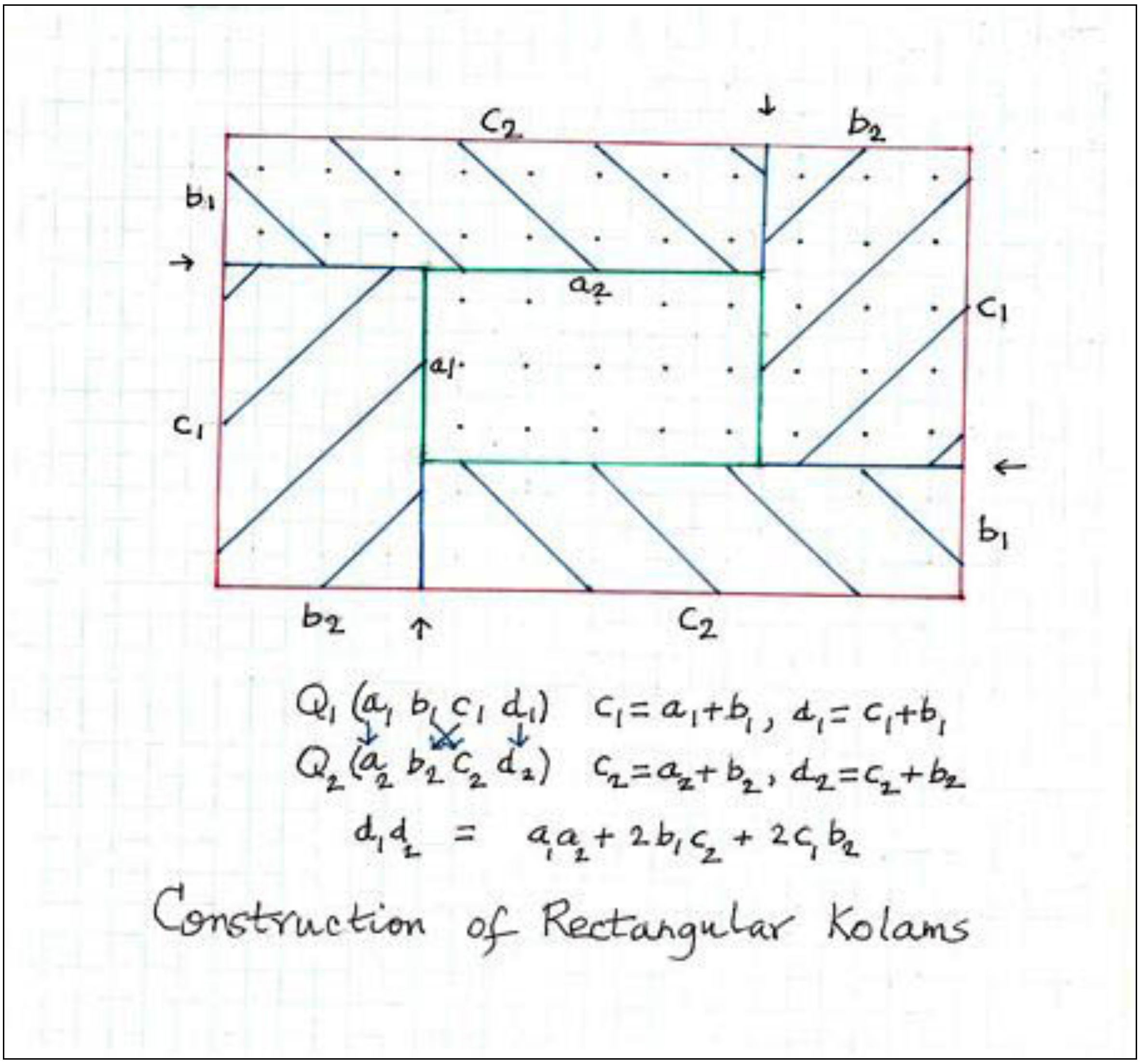

Figure 7: 9 x 13

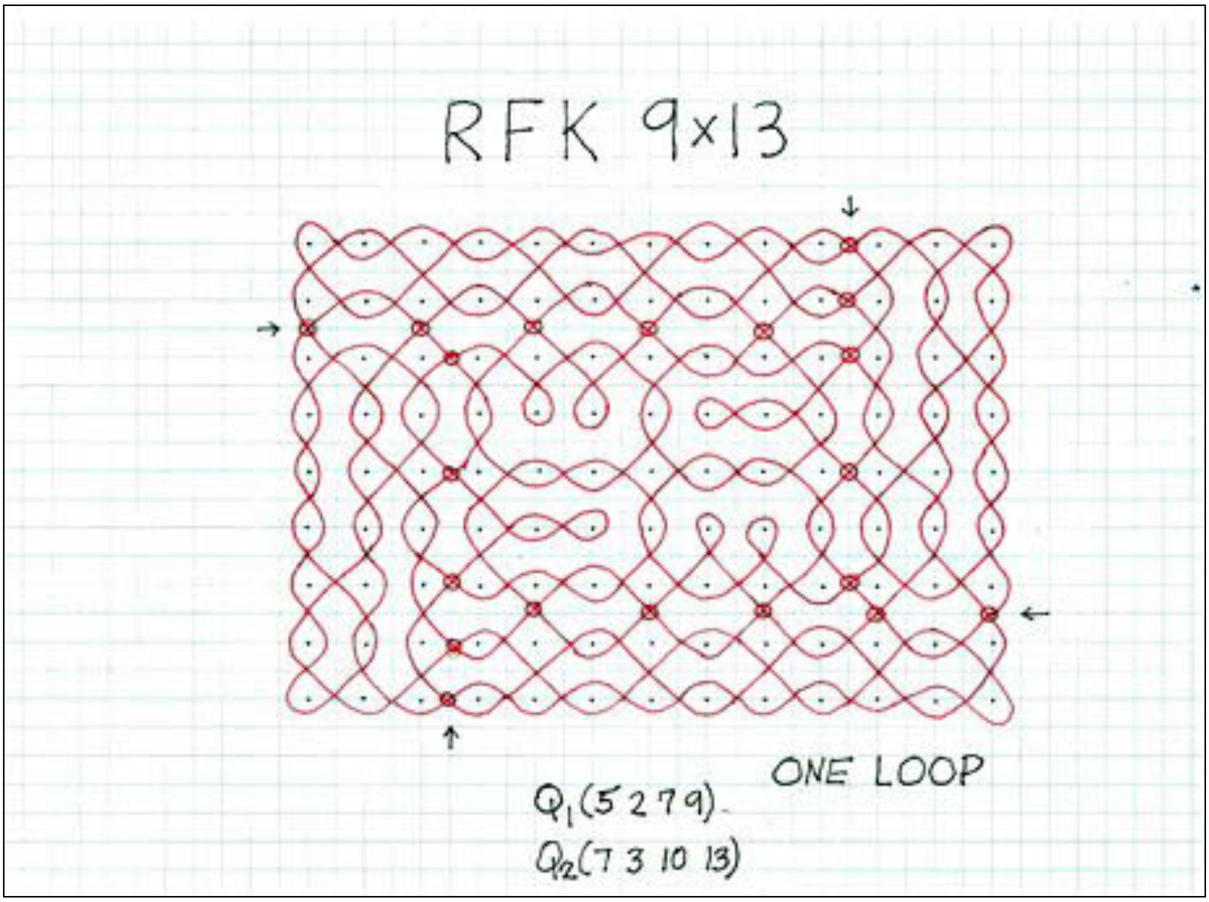

Figure 8: 15^2